Distinct zeta zeros on the critical line

Josu San Martin · 21 November 2026 · Lean source

Abstract

This note checks in Lean that distinct zeros on the critical line account asymptotically for at least 669/800 (83.625%) of the nontrivial zeros of the Riemann zeta function, counted with multiplicity. The Lean proof covers dyadic and cumulative windows. A cubic trace argument gives a stronger analytic bound of 85.0881905187…% under RH and the window estimates on page 3. The zeta theorem at that stronger constant still awaits end-to-end formal verification.

The Lean theorem

For a window $I = (a, b]$, write $N(I)$ for the number of nontrivial zeta zeros with imaginary part in I, counted with multiplicity. Let $D(I)$ count distinct locations in the strip and $L(I)$ count distinct locations on the line $\mathrm{Re}(s) = ½$.

> For every $\varepsilon > 0$ there is a height T_0 such that for every $T \geq T_0$, both $I = (T, 2T]$ and $I = (0, T]$ satisfy
>
> $$L(I) \geq (669/800 - \varepsilon)\, N(I).$$

Proof

The distinct-zero theorem gives $D(I) \geq (c - \varepsilon)N(I)$ in each window, where c is the Montgomery–Taylor distinct-zero constant. A rational Taylor enclosure proves $669/800 \leq c$. Since $N(I)$ is nonnegative, the same inequality holds with 669/800 in place of c. [1, 5]

Apply the published Lean theorem `riemannHypothesis` to the Zeta23 counting identity. Every nontrivial zero in the window lies on the critical line, so $D(I) = L(I)$. Substituting this identity gives the claimed inequality. Take the larger of the two height thresholds to obtain both bounds simultaneously. [1, 2]

Keeping the second trace in the cubic bound

Alpöge and Furman use a third trace under RH to obtain a cumulative distinct-zero proportion of 0.85082… [1, §7.3]. A subsequent rational-window calculation gives 0.850828702939… [7]. The following variation keeps the second trace as a separate term.

Let H be a positive semidefinite Hermitian Gram matrix indexed by the D distinct zero locations. Its diagonal entries are their positive integer multiplicities m_{j}. Write $N = \Sigma m_{\mathrm{j}}$, S for the number of entries equal to one, and $R_{\mathrm{k}} = \mathrm{tr}(H^{\mathrm{k}})$.

> For $0 \le a \le 1$,
>
> $$(18 - 12a)D \ge (9 - 10a)N + 2(1 + a)R_2 - R_3 + (8 - 4a - a^2)S.$$

Proof of the finite inequality

Set $f(x) = x^3 - 2(1 + a)x^2$. For each diagonal index, use the spectral measure of its unit coordinate vector. This is a probability measure on $x \ge 0$ with mean $m = m_{\mathrm{j}}$. Its integral of f is the corresponding diagonal entry of $f(H)$.

When $m = 1$, the identity $f(x) + (1 + a)^2 x = x(x - 1 - a)^2$ gives an integral of at least $-(1 + a)^2$. For $m \ge 2$ the tangent line at m lies below f throughout the spectrum, since

$$f(x) - f(m) - f'(m)(x - m) = (x - m)^2(x + 2m - 2 - 2a) \ge 0.$$

Integration gives at least $f(m)$. Subtracting $(9 - 10a)m - (18 - 12a)$ leaves $(m - 2)(m - 3)(m + 3 - 2a)$, nonnegative at every integer $m \ge 2$. Summing over indices gives the displayed inequality.

Evaluating the constant

Choose $a = 37/1000$. On $|t| \le ½$ use the frequency window $v(t) = 1 - (743/500)t^2 + (1197/1000)t^4$, extended by zero outside. Normalize its Fourier transform by $\int v$ so the Gram kernel equals one at the origin. The sine-kernel moment integrals in [1, 7, 8] then give the following normalized trace values:

$$r_2 = 4677629393520 \,/\, 3522041957357,$$
$$r_3 = 1484618890495233180 \,/\, 753262635461898947.$$

The analytic inputs on a given window are $R_k/N \to r_k$ for $k = 2, 3$ and $\liminf S/N \ge 19/27$. Since the coefficient of S is positive, insertion in the finite inequality gives

$$\liminf D/N \ge C_* = 92064269160344668763683 51 \,/\, 10819864495774716474708000$$
$$= 0.8508819051874155\ldots > 8508819 \,/\, 10000000.$$

Under RH the same estimate holds for L/N. I reproduced the fraction by exact integration of the quartic window. Exact-arithmetic script.

Why the interval matters

The written argument [10] uses the cumulative simple-zero estimate of Bui and Heath-Brown [6]. For $(T, 2T]$ it uses the same mollifier at the two endpoints and subtracts the underlying moment sums before applying Cauchy–Schwarz. The correlation step uses restricted-support pair and triple statistics [8], with smooth windows of width below one followed by a limiting argument. These are the analytic steps needed to apply the constant to both windows. The continuation reports kernel replay for the finite matrix inequality; the stronger zeta bound as a whole has no corresponding replay in this note.

What would suffice for 100%

For $0 < u < 1$, define the mean of the square root of a nearby-shift ratio, taking the height limit first:

$$F(u) = \limsup_{T\to\infty} (1/T) \int_T^{2T} |\zeta(½ + 1/\log T + it) \,/\, \zeta(½ + u/\log T + it)|^{½} \, dt.$$

A zero of multiplicity m contributes $m(m-1)/2$ coincident pairs to the product estimate in [10]. The proposed route to density one requires $F(u) = o(\log(1/u))$ as $u \downarrow 0$. Rodgers' bound for fixed shifts [9] allows its constant to depend on u. The required upper estimate remains open.

Reproducing the 669/800 proof

The build uses Lean 4.33.1 and the Mathlib revision of the published RH development. The required Zeta23 source modules were rebuilt on that revision. `DistinctBound.lean` checks the rational comparison. `CriticalLine669.lean` imports that module and the RH release's `Main` module, then proves `dyadic_669` and `cumulative_669`.

Lean's kernel replayed both new modules against their imports. The final theorems take no RH argument. Their axiom reports list only `propext`, `Classical.choice` and `Quot.sound`. A fresh Zeta23 checkout reproduced the reports using the packaged script. The local runs used hash-checked published RH artifacts. Sutherland reports separate verification of the underlying RH development. [3]

Download the source and build script or inspect the verification log. The README gives the exact repository revisions and setup.

Scope and attribution

Chirre, Gonçalves and de Laat proved under RH that at least 84.77% of zeros up to height T are distinct asymptotically [4, Corollary 3]. Their corollary concerns $(0, T]$. Lower bounds on the counts up to T and $2T$ cannot simply be subtracted to obtain the same proportion in $(T, 2T]$. The joint build checks 83.625% in both windows. The larger cubic constant uses a further analytic step at each window.

The rational enclosure uses a Taylor estimate from typh's *five-point-pyramid* proof [5], which credits Samuel Lavery. The source includes the attribution and Apache 2.0 license. I used Codex and ChatGPT to develop, test and write this note.

References

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3. Andrew Sutherland. Verification files for the Lean proof of the Riemann hypothesis.
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7. teal-sea. Optimizing the cubic window. Rational comparison window and trace functionals.
8. Z. Rudnick and P. Sarnak. Zeros of principal L-functions and random matrix theory.
9. B. Rodgers. Tail bounds for counts of zeros and eigenvalues, and an application to ratios.
10. Shared research continuation, 21 November 2026. Shifted cubic inequality, dyadic argument and nearby-shift ratio criterion.